Near-exact formulation of transport theory

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Two-particle collisions

Before collision: $\vec{v}_1, \vec{v}_2$ After collision: $\vec{v}_1', \vec{v}_2'$

Conservation of momentum: $m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1^{'} + m_2\vec{v}_2^{'} = \vec{P} = \text{constant}$

(No extern forces)

Velocity of the center of mass: $\vec{c} \equiv \dfrac{\vec{P}}{m_1 + m_2} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$

Relative velocity: $\vec{V} = \vec{v}_1 - \vec{v}_2$ Reduced mass: $\mu \equiv \dfrac{m_1 m_2}{m_1 + m_2}$

Total kinetic energy: $K = \dfrac{1}{2} m_1 \vec{v}_1^2 + \dfrac{1}{2} m_2 \vec{v}_2^2 = \dfrac{1}{2}(m_1 + m_2)\vec{c}^{\,2} + \dfrac{1}{2}\mu\vec{V}^2$

Conservation of energy: $K = K' \longrightarrow \left|\vec{V}\right| = \left|\vec{V}'\right|$

(Elastic collision)

Scattering cross sections

$$\sigma'(\vec{v}_1, \vec{v}_2 \to \vec{v}_1', \vec{v}_2') d^3\vec{v}_1' d^3\vec{v}_2' :$$

The number of molecules per unit time (unit flux of type A_1 molecules incident with relative velocity $\vec{V}$ upon a type A_2 molecules) emerging after scattering with respective final velocities between $\vec{v}_1'$ and $\vec{v}_1' + d\vec{v}_1'$ and between $\vec{v}_2'$ and $\vec{v}_2' + d\vec{v}_2'$

Scattering cross sections

Two symmetry properties:

(1) t -> -t: $\sigma'(\vec{v}_1, \vec{v}_2 \to \vec{v}_1', \vec{v}_2') d^3\vec{v}_1' d^3\vec{v}_2' = \sigma'(-\vec{v}_1', -\vec{v}_2' \to -\vec{v}_1, -\vec{v}_2) d^3\vec{v}_1 d^3\vec{v}_2$

$d^3\vec{v}_1 d^3\vec{v}_2 = d^3\vec{v}_1' d^3\vec{v}_2' \longrightarrow \sigma'(\vec{v}_1, \vec{v}_2 \to \vec{v}_1', \vec{v}_2') = \sigma'(-\vec{v}_1', -\vec{v}_2' \to -\vec{v}_1, -\vec{v}_2)$

(2) **r** -> -**r**:

$$\sigma'(\vec{v}_1, \vec{v}_2 \to \vec{v}_1', \vec{v}_2') = \sigma'(-\vec{v}_1, -\vec{v}_2 \to -\vec{v}_1', -\vec{v}_2')$$

When t -> -t and **r** -> -**r**:

$$\sigma'(\vec{v}_1, \vec{v}_2 \to \vec{v}_1', \vec{v}_2') = \sigma'(\vec{v}_1', \vec{v}_2' \to \vec{v}_1, \vec{v}_2)$$

Boltzmann equation

Volume element: $d^3\vec{r} \sim (\vec{r}, \vec{r} + d\vec{r})$ Time: $dt \sim (t, t + dt)$

$$D_c f = -D_c^{(-)} f + D_c^{(+)} f$$

$$\mathrm{A} : (\vec{v}, \vec{v} + d\vec{v})$$

$$D_c^{(+)} f \longrightarrow \qquad \longrightarrow D_c^{(-)} f$$

$$\mathrm{A}_1 : (\vec{v}_1, \vec{v}_1 + d\vec{v}_1)$$

$\mathrm{B} : (\vec{v}', \vec{v}' + d\vec{v}')$

$\mathrm{B}_1 : (\vec{v}_1', \vec{v}_1' + d\vec{v}_1')$

$\mathrm{B} : (\vec{v}', \vec{v}' + d\vec{v}')$

$\mathrm{B}_1 : (\vec{v}_1', \vec{v}_1' + d\vec{v}_1')$

Boltzmann equation

$$\vec{v}, \vec{v}_1 \rightarrow \vec{v}', \vec{v}_1' :$$

$$D_c^{(-)} f(\vec{r}, \vec{v}, t) d^3\vec{r} d^3\vec{v} d^3 t =$$

$$\int_{\vec{v}_1'} \int_{\vec{v}'} \int_{\vec{v}_1} [|\vec{v} - \vec{v}_1| f(\vec{r}, \vec{v}, t) d^3\vec{v}][f(\vec{r}, \vec{v}_1, t) d^3\vec{r} d^3\vec{v}_1][\sigma'(\vec{v}, \vec{v}_1 \rightarrow \vec{v}', \vec{v}_1') d^3\vec{v}' d^3\vec{v}_1']$$

$|\vec{v} - \vec{v}_1| f(\vec{r}, \vec{v}, t) d^3\vec{v}$ Relative flux of A molecules incident upon an A_1 molecule

$f(\vec{r}, \vec{v}_1, t) d^3\vec{r} d^3\vec{v}_1$ The total number of A_1 molecules in this volume element

$\sigma'(\vec{v}, \vec{v}_1 \rightarrow \vec{v}', \vec{v}'_1) d^3\vec{v}' d^3\vec{v}'_1$ The scattering probability

Boltzmann equation

$$\vec{v}', \vec{v}_1' \to \vec{v}, \vec{v}_1 :$$

$$D_c^{(+)} f(\vec{r}, \vec{v}, t) d^3\vec{r} d^3\vec{v} d^3 t =$$

$$\int_{\vec{v}_1'} \int_{\vec{v}'} \int_{\vec{v}_1} [|\vec{v}' - \vec{v}_1'| f(\vec{r}, \vec{v}', t) d^3\vec{v}'][f(\vec{r}, \vec{v}_1', t) d^3\vec{r} d^3\vec{v}_1'][\sigma'(\vec{v}', \vec{v}_1' \to \vec{v}, \vec{v}_1) d^3\vec{v} d^3\vec{v}_1]$$

$|\vec{v}' - \vec{v}'_1| f(\vec{r}, \vec{v}', t) d^3\vec{v}'$ Relative flux of B molecules incident upon an B_1 molecule

$f(\vec{r}, \vec{v}'_1, t) d^3\vec{r} d^3\vec{v}'_1$ The total number of B_1 molecules in this volume element

$\sigma'(\vec{v}', \vec{v}'_1 \to \vec{v}, \vec{v}_1) d^3\vec{v} d^3\vec{v}_1$ The scattering probability

Boltzmann equation

$$\sigma'(\vec{v},\vec{v}_1 \to \vec{v}',\vec{v}_1') = \sigma'(\vec{v}',\vec{v}_1' \to \vec{v},\vec{v}_1)$$

$$D_c f = \int_{\vec{v}_1'}\int_{\vec{v}'}\int_{\vec{v}_1} (f'f_1' - ff_1) V \sigma'(\vec{v},\vec{v}_1 \to \vec{v}',\vec{v}_1') d^3\vec{v}_1 d^3\vec{v}' d^3\vec{v}_1'$$

Boltzmann equation:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + \frac{\vec{F}}{m}\frac{\partial f}{\partial \vec{v}} = \int_{\vec{v}_1'}\int_{\vec{v}'}\int_{\vec{v}_1} (f'f_1' - ff_1) V \sigma'(\vec{v},\vec{v}_1 \to \vec{v}',\vec{v}_1') d^3\vec{v}_1 d^3\vec{v}' d^3\vec{v}_1'$$

Thank you!