Fundamental of Statistical and Thermal Physics by F. Reif
Chapter 13
Transport Theory Using the Relaxation-time Approximation

Jia Le MA

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Transport theory in this Chapter

- Take the distribution of molecular velocity into account
- Still use drastic approximations

Some definition

$f\left(r,v,t\right)$ Molecular distribution function

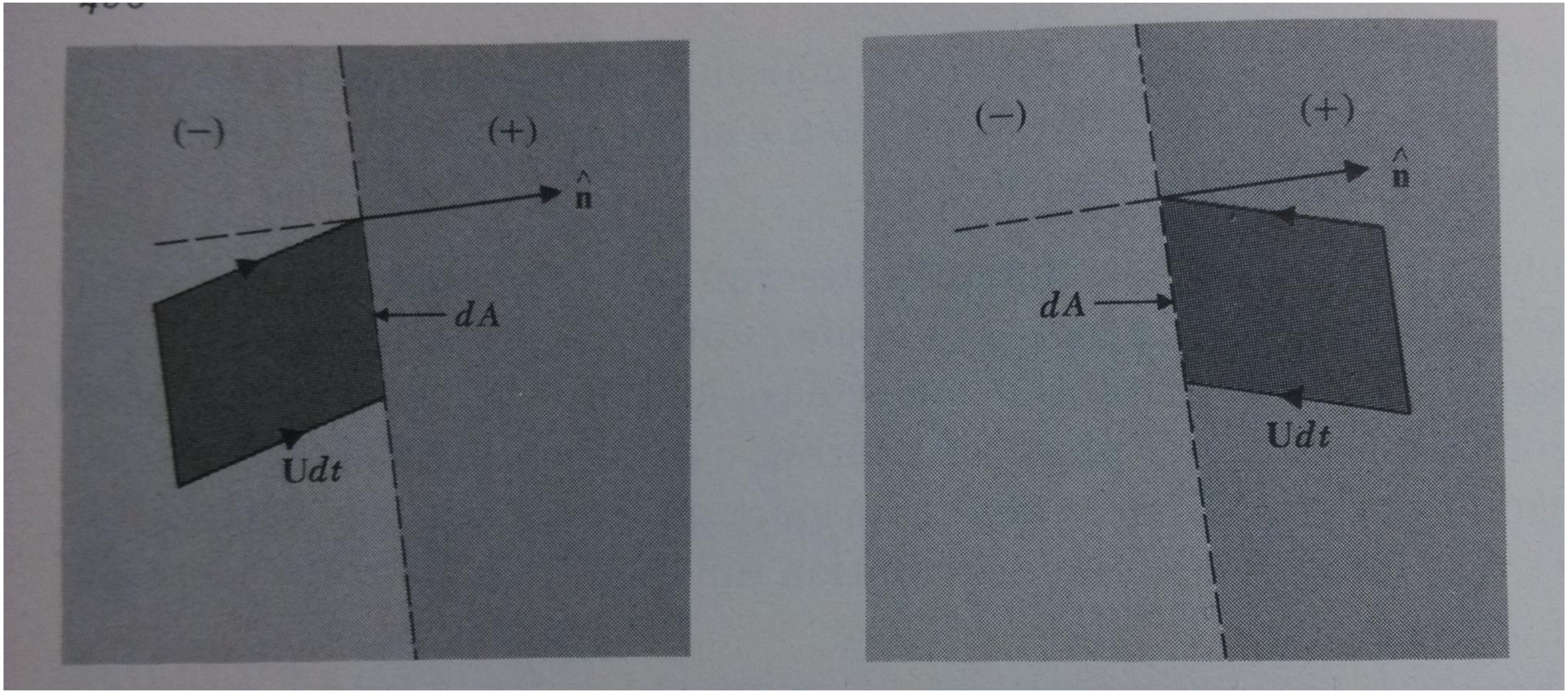
$f\left(r,v,t\right)d^3rd^3v \equiv$ Mean number of molecules between r and r + dr; v and v + dv

$n\left(r,t\right)=\int d^3vf\left(r,v,t\right)$ Mean number

$\langle\chi(\mathbf{r},t)\rangle \equiv \bar{\chi}(\mathbf{r},t) \equiv \frac{1}{n(\mathbf{r},t)}\int d^3\mathbf{v}\, f(\mathbf{r},\mathbf{v},t)\chi(\mathbf{r},\mathbf{v},t)$ Mean function

$u\left(r,t\right)\equiv\left\langle v\left(r,t\right)\right\rangle=\frac{1}{n\left(r,t\right)}\int d^3vf\left(r,v,t\right)v$ Mean v

$U\equiv v-u \qquad \left\langle U\right\rangle\equiv\left\langle v\right\rangle-u=0$ Peculiar velocity



$\hat{n}\cdot U > 0$ $\hat{n}\cdot U < 0$

(-) to (+) $\int_{\hat{n}\cdot U>0} f(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{v}\, |\hat{\mathbf{n}}\cdot \mathbf{U}\, dt\, dA|\, \chi(\mathbf{r},\mathbf{v},t)$

(+) to (-) $\int_{\hat{n}\cdot U<0} f(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{v}\, |\hat{\mathbf{n}}\cdot \mathbf{U}\, dt\, dA|\, \chi(\mathbf{r},\mathbf{v},t)$

The net flux

$$\mathfrak{F}_n(\boldsymbol{r},t) = \int_{\hat{\boldsymbol{n}}\cdot\boldsymbol{U}>0} d^3\boldsymbol{v}\, f|\hat{\boldsymbol{n}}\cdot\boldsymbol{U}|\,\chi - \int_{\hat{\boldsymbol{n}}\cdot\boldsymbol{U}<0} d^3\boldsymbol{v}\, f|\hat{\boldsymbol{n}}\cdot\boldsymbol{U}|\,\chi$$

$$\mathfrak{F}_n(\boldsymbol{r},t) = \int d^3\boldsymbol{v}\; f\hat{\boldsymbol{n}}\cdot\boldsymbol{U}\chi$$

$$n = \int d^3 v f$$

$$\mathfrak{F}_n(\boldsymbol{r},t) = n\,\langle\hat{\boldsymbol{n}}\cdot\boldsymbol{U}\chi\rangle$$

$$\mathfrak{F}_n = \hat{\boldsymbol{n}}\cdot\boldsymbol{\mathfrak{F}}$$

$$\boldsymbol{\mathfrak{F}} = n\langle\boldsymbol{U}\chi\rangle$$

Example of gas viscosity

$$\chi = mv_\alpha \qquad \hat{n}\cdot U = U_z$$

$$P_{z\alpha} = nm\langle U_z v_\alpha\rangle$$

$$= nm\langle U_z(u_\alpha + U_\alpha)\rangle = nm[u_\alpha\langle U_z\rangle + \langle U_z U_\alpha\rangle]$$

$$P_{z\alpha} = nm\langle U_z U_\alpha\rangle$$

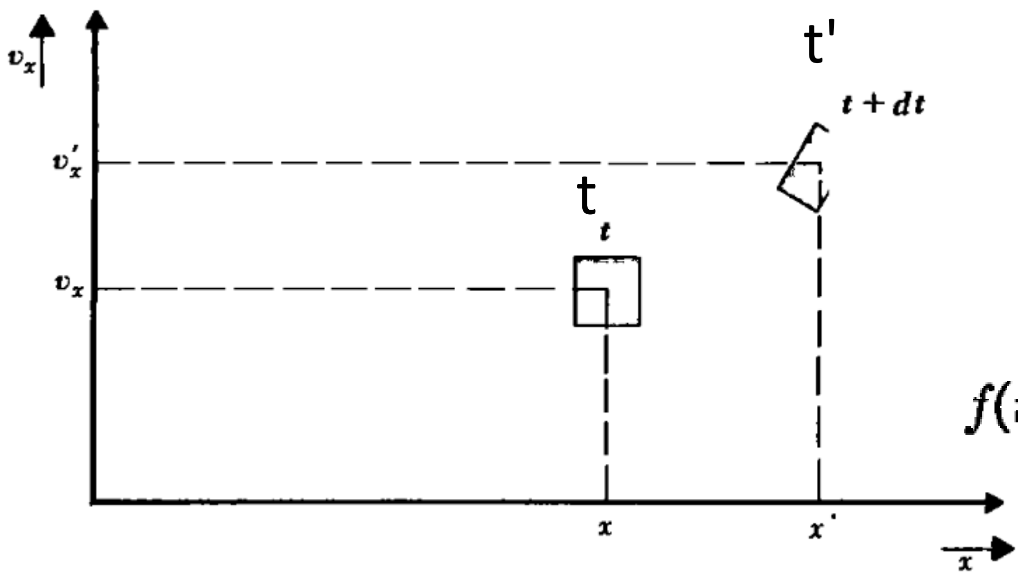
Since $u_x \neq 0$ and $u_z = 0$

$$P_{z\alpha} = nm\langle v_z v_\alpha\rangle = m\int d^3v\, f v_z v_\alpha$$

Transport without collision

For each molecule: mass m, external force F(r,t)

F(r,t) here do not depend on *v*



At time t' = t +dt

$$\boldsymbol{r}' = \boldsymbol{r} + \dot{\boldsymbol{r}}\,dt = \boldsymbol{r} + \boldsymbol{v}\,dt$$

$$\boldsymbol{v}' = \boldsymbol{v} + \dot{\boldsymbol{v}}\,dt = \boldsymbol{v} + \frac{1}{m}\boldsymbol{F}\,dt$$

In absence of collision,

$$f(\boldsymbol{r}',\boldsymbol{v}',t')\,d^3\boldsymbol{r}'\,d^3\boldsymbol{v}' = f(\boldsymbol{r},\boldsymbol{v},t)\,d^3\boldsymbol{r}\,d^3\boldsymbol{v}$$

Which means…

The distortion of phase space

$$d^3\mathbf{r}'\, d^3\mathbf{v}' = |J|\, d^3\mathbf{r}\, d^2\mathbf{v}$$

$$\frac{\partial x_\alpha'}{\partial x_\gamma} = \delta_{\alpha\gamma}; \qquad \frac{\partial x_\alpha'}{\partial v_\gamma} = \delta_{\alpha\gamma}\, dt$$

$$\frac{\partial v_\alpha'}{\partial x_\gamma} = \frac{1}{m}\frac{\partial F_\alpha}{\partial x_\gamma}\, dt; \qquad \frac{\partial v_\alpha'}{\partial v_\gamma} = \delta_{\alpha\gamma}$$

$$J = \frac{\partial(x', y', z', v_x', v_y', v_z')}{\partial(x, y, z, v_x, v_y, v_z)} = \left|\begin{array}{ccc:ccc} 1 & 0 & 0 & dt & 0 & 0 \\ 0 & 1 & 0 & 0 & dt & 0 \\ 0 & 0 & 1 & 0 & 0 & dt \\ \hdashline \frac{1}{m}\frac{\partial F_x}{\partial x}\, dt & \cdots & \cdots & 1 & 0 & 0 \\ \cdots & \cdots & \cdot & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 & 1 \end{array}\right|$$

$$J = 1 + \mathcal{O}(dt^2) \qquad d^3\mathbf{r}'\, d^3\mathbf{v}' = d^3\mathbf{r}\, d^3\mathbf{v}$$

Boltzmann equation without collisions

$$f(\mathbf{r}',\mathbf{v}',t')\, d^3\mathbf{r}'\, d^3\mathbf{v}' = f(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{r}\, d^3\mathbf{v} \qquad d^3\mathbf{r}'\, d^3\mathbf{v}' = d^3\mathbf{r}\, d^3\mathbf{v}$$

Then, $f(\mathbf{r}',\mathbf{v}',t') = f(\mathbf{r},\mathbf{v},t)$ f remain unchanged

$$f(\mathbf{r} + \dot{\mathbf{r}}\, dt,\, \mathbf{v} + \dot{\mathbf{v}}\, dt,\, t + dt) - f(\mathbf{r},\mathbf{v},t) = 0$$

By partial derivatives,

$$\left[\left(\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z}\right) + \left(\frac{\partial f}{\partial v_x}\dot{v}_x + \frac{\partial f}{\partial v_y}\dot{v}_y + \frac{\partial f}{\partial v_z}\dot{v}_z\right) + \frac{\partial f}{\partial t}\right] dt = 0$$

$$Df = 0$$

$$Df \equiv \frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{r}} + \dot{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}}$$

When collision take place

Number of molecules change in $d^3r\ d^3v$

Net increase per time in $d^3r\ d^3v$: $D_C f\ d^3r\ d^3v$

$$f(\boldsymbol{r} + \dot{\boldsymbol{r}}\,dt,\ \boldsymbol{v} + \dot{\boldsymbol{v}}\,dt,\ t + dt)\ d^3\boldsymbol{r}'\ d^3\boldsymbol{v}' = f(\boldsymbol{r},\boldsymbol{v},t)\ d^3\boldsymbol{r}\ d^3\boldsymbol{v} + D_C f\ d^3\boldsymbol{r}\ d^3\boldsymbol{v}\ dt$$

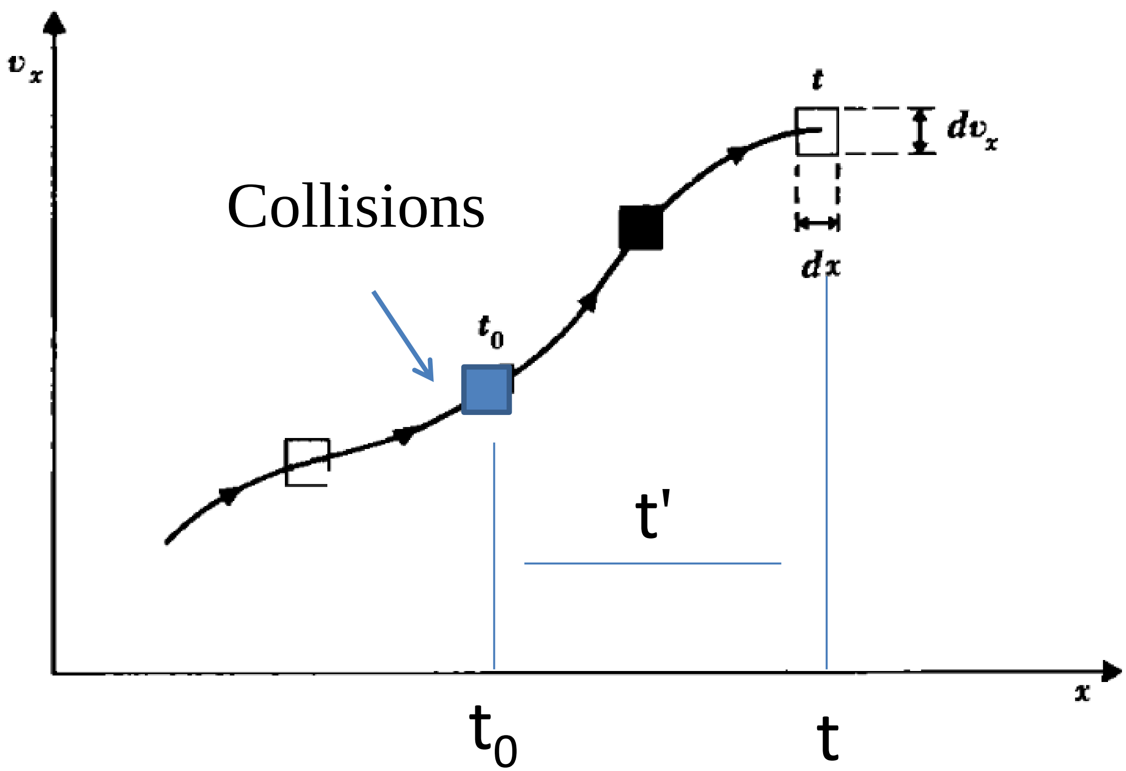
$$Df = D_C f$$ **Boltzmann Equation**

Assume: effect of collisions is to restore local equilibrium situation $f^{(0)}$. Then disturb it, we get f. Relaxation time: τ_0

$$D_C f = -\frac{f - f^{(0)}}{\tau_0}$$

$$Df \equiv \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = -\frac{f - f^{(0)}}{\tau_0}$$

Path integral formulation



If no collisions

$$f(\boldsymbol{r},\boldsymbol{v},t) = f(\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)$$

Collision between t_0 and t_0-dt';
continue to move without further collision for t'

Such probability:

$$\frac{dt'}{\tau}\, e^{-t'/\tau}$$

Number of molecules survive:

$$f(\boldsymbol{r},\boldsymbol{v},t) = \int_0^\infty f^{(0)}(\boldsymbol{r}_0,\, \boldsymbol{v}_0,\, t - t')\, e^{-t'/\tau}\, \frac{dt'}{\tau}$$

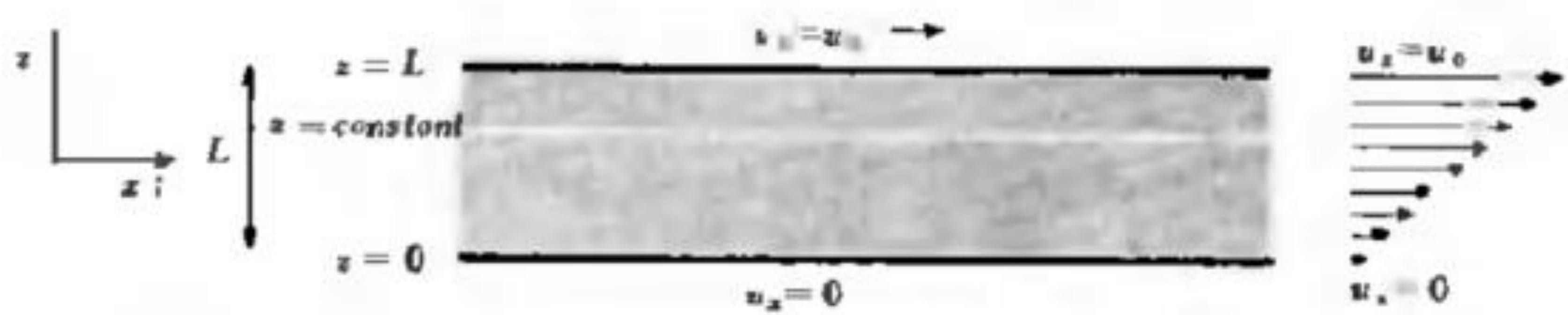
Equivalent to
Boltzmann equation

Calculation of Viscosity

$$f^{(0)}(\mathbf{r},\boldsymbol{v},t) = g[v_x - u_x(z),\, v_y,v_z] = g(U_x,U_y,U_z)$$

where $\quad U_x = v_x - u_x(z), \qquad U_y = v_y, \qquad U_z = v_z$

$$g(U_x,U_y,U_z) = g(U) = n\left(\frac{m\beta}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{1}{2}\beta m U^2}$$



From Boltzmann equation:

$$Df \equiv \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = -\frac{f - f^{(0)}}{\tau_0}$$

f does not depend on t; no F

$$v_z \frac{\partial f}{\partial z} = -\frac{f - f^{(0)}}{\tau}$$

Assume $\partial f / \partial z$ sufficiently small

$$f = f^{(0)} + f^{(1)}, \qquad \text{with } f^{(1)} \ll f^{(0)}$$

$$f^{(1)} = f - f^{(0)} = -\tau v_z \frac{\partial f^{(0)}}{\partial z} = \tau v_z \frac{\partial g}{\partial U_x} \frac{\partial u_x}{\partial z}$$

$$P_{zx} = m \int d^3\boldsymbol{v}\, f U_z U_x$$

Where $\int d^3\boldsymbol{v}\, f^{(0)} U_z U_x = 0$

By definition,

$$P_{zx} = -\eta \frac{\partial u_x}{\partial z}$$

Then,

$$\eta = -m \int d^3\boldsymbol{U} \frac{\partial g}{\partial U_x} U_z{}^2 U_x \tau$$

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Substitute τ by a mean $\bar{\tau}$,
After integrating,

$$\eta = m\bar{\tau} \int d^3\boldsymbol{U}\, g U_z{}^2 = m\bar{\tau} n \overline{U_z{}^2}$$

In Maxwell distribution, $\overline{\frac{1}{2} m U_z{}^2} = \frac{1}{2} kT$ $\eta = nkT\bar{\tau}$

Or $\overline{U_z{}^2} = \frac{1}{3}\overline{U^2}$ $\eta = \frac{1}{3} nm\bar{\tau}\overline{U^2}$

If one put $\overline{U^2} \approx \bar{U}^2$ Then, $\tau\bar{U} = l$

At the same time u_x is small so that $\bar{U} \approx \bar{v}$

$\eta = \frac{1}{3} nml\bar{v}$ which is equal to the result of elementary kinetic theory