Chapter 11: Magnetism and low temperatures

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Outline

1. **Terminologies**
2. **Magnetic work**
3. **Low temperatures cooling**
   - Mechanical cooling
   - Magnetic cooling

- In this report, the units system is chosen to be Gaussian Units, which is common used in the field of electrodynamics.
- Please see an example first.
- In qualitative study, Gaussian Units will not produce big differences from SI Units.
- However, in quantitative study, a difference of a constant between them does really make sense.

For example, the unit of charge in SI system is *Coulomb*. Then, Coulomb's law is presented as

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}, \tag{1.1}$$

where ε_0 is the vacuum permittivity. As a comparison, in terms of Gaussian system, the unit of charge is $statC$, which has a dimension of $g^{1/2}cm^{3/2}s^{-1}$. The Coulomb's law can be expressed more simply like

$$F = \frac{Q_1 Q_2}{r^2}. \tag{1.2}$$

Charge density

At a point $\mathbf{r}$,

$$\rho(\mathbf{r}) = \frac{1}{c} \lim_{V \to 0} \frac{1}{V(\mathbf{r})} \int_{V(\mathbf{r})} q(x, y, z) dx dy dz. \tag{1.3}$$

An approximate *mean charge density over* V,

$$\rho_V(\mathbf{r}) := \frac{1}{cV} \sum_{(x,y,z) \in V} q(x, y, z). \tag{1.4}$$

At one point,

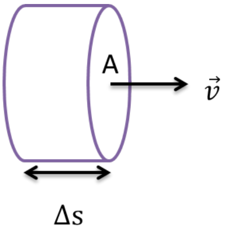
$$\mathbf{J}(\mathbf{r}) := \frac{1}{c} \lim_{V \to 0} \frac{1}{V(\mathbf{r})} \int_{V(\mathbf{r})} q(x,y,z)\mathbf{v}(x,y,z)dxdydz. \tag{1.5}$$

An approximate *mean current density over* V,

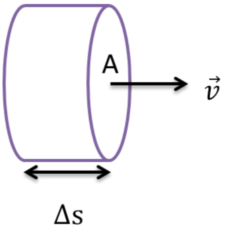
$$\mathbf{J}_V(\mathbf{r}) := \frac{1}{cV} \sum_{(x,y,z)\in V} q(x,y,z)\mathbf{v}(x,y,z), \tag{1.6}$$

which is more usually used than the above one.

A simple example



- Suppose the charges are flowing rightward with a same velocity $\mathbf{v}$, $v = |\mathbf{v}|$, the charge density is ρ and the current density is J everywhere.



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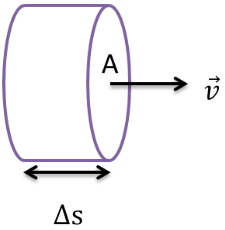

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- If it takes Δt to pass a Δs long distance. Our aim is to show $I = JA$, where $I = Q/\Delta t$.
- Clearly, $V = A\Delta s$, $Q = \rho V = \rho A \Delta s$ and $\Delta t = \Delta s / v$.

$$\Rightarrow J = \frac{Qv}{V} = \frac{A\Delta l \rho v}{A\Delta s} = \rho v.$$

$$\Rightarrow I = \frac{Q}{t} = \frac{A\Delta\rho}{\Delta t} = \rho v A = JA.$$

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- Now we call the new vector field as *magnetic field*, sometimes *magnetic induction* instead, for historical reasons.
- *Magnetic induction*, in Gaussian units,

$$\mathbf{B}(\mathbf{r}) := \frac{1}{c}\mathbf{v} \times \mathbf{E}(\mathbf{r}),$$

at $\mathbf{r}$, which, together with $\mathbf{E}, \mathbf{B}, \mathbf{v}$, is observed in the same frame of reference.

At one point $\mathbf{r}$, the *magnetic moment* is defined as

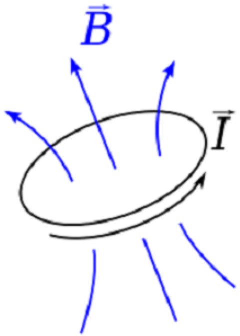
$$\mu(\mathbf{r}) := \frac{1}{2c}\mathbf{r} \times \mathbf{J}(\mathbf{r}). \tag{1.7}$$

For a charge distribution within a volume V, it is defined as

$$\mathbf{m} := \frac{1}{2c}\int_V \mathbf{r} \times \mathbf{J}(\mathbf{r}) dxdydz, \tag{1.8}$$

wherein the integral, $\mathbf{r} = (x, y, z)$.

- Imagine a particle with a charge $+q$ rotating around an axis with a constant angular speed ω. Suppose the rotating radius is R and the velocity at the position $\mathbf{r}$ is $\mathbf{v}$, $v = |\mathbf{v}| = \omega R$ keep constant.



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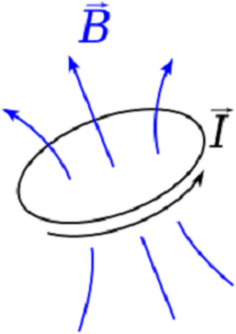

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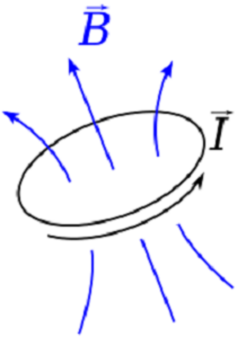
- As a conclusion,

$$\mathbf{m} = \frac{1}{2} q \mathbf{R} \times \mathbf{v} = I \mathbf{A}. \qquad (1.9)$$



Magnetic field strength

- *Magnetization* is defined as magnetic moment per volume,

$$\mathbf{M} = \frac{\mathbf{m}}{V},$$

  where $\mathbf{m}$ is the magnetic moment of the system.
- The *magnetic field strength* is then defined to be

$$\mathbf{H} := \mathbf{B} - 4\pi\mathbf{M}.$$

Other magnetic relevant quantities

- *Magnetic susceptibility*

$$\chi := \frac{|\mathbf{M}|}{|\mathbf{H}|}$$

- *Magnetic permeability*

$$\mu := 1 + 4\pi\chi$$

- These are two dimensionless quantity. They both are relevant to the materials' properties.
- Relations with **B** and **H**.

$$\mathbf{B} = 4\pi\mathbf{M} + \mathbf{H} = 4\pi\chi\mathbf{H} + \mathbf{H} = \mu\mathbf{H}.$$

Outline

Magnetic force

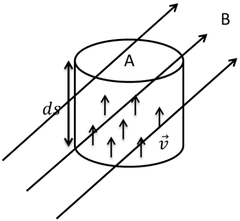
- Suppose a charge q with a velocity $\mathbf{v}$ located near some moving source charges.

Magnetic force

- Suppose a charge q with a velocity $\mathbf{v}$ located near some moving source charges.
- That vector field $\mathbf{B}$ is called magnetic induction while that force is called *magnetic force*, or sometimes named after *Lorentz*, which by deduction should be

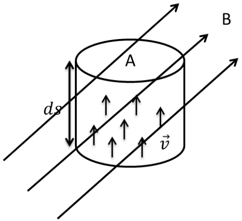
$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}.$$

Magnetic force



- For a segment of current loop ds loading an electronic current of **I** in a magnetic field **B**, suppose a magnetic force **F** on it.

Magnetic force



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- Indeed, the current loop can be treated as a number of charged particles flowing with a velocity $\mathbf{v}$. Then,

$$d\mathbf{F} = \rho A ds \mathbf{v} \times \mathbf{B}. \tag{2.1}$$

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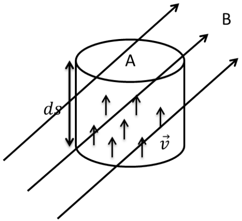

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$$d\mathbf{F} = \rho A ds \mathbf{v} \times \mathbf{B}. \qquad (2.1)$$

- Since $\rho\mathbf{v} = \mathbf{J}$ and $\mathbf{J}A = \mathbf{I}$, it is clear that

$$d\mathbf{F} = \mathbf{I} \times \mathbf{B} ds. \qquad (2.2)$$

Integrate both sides on the whole loop L, then we have

$$\mathbf{F} = \int_L \mathbf{I} \times \mathbf{B} ds. \qquad (2.3)$$

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- Move the sample towards x direction by a length of dx. Then the work done by the magnetic field on the sample should be

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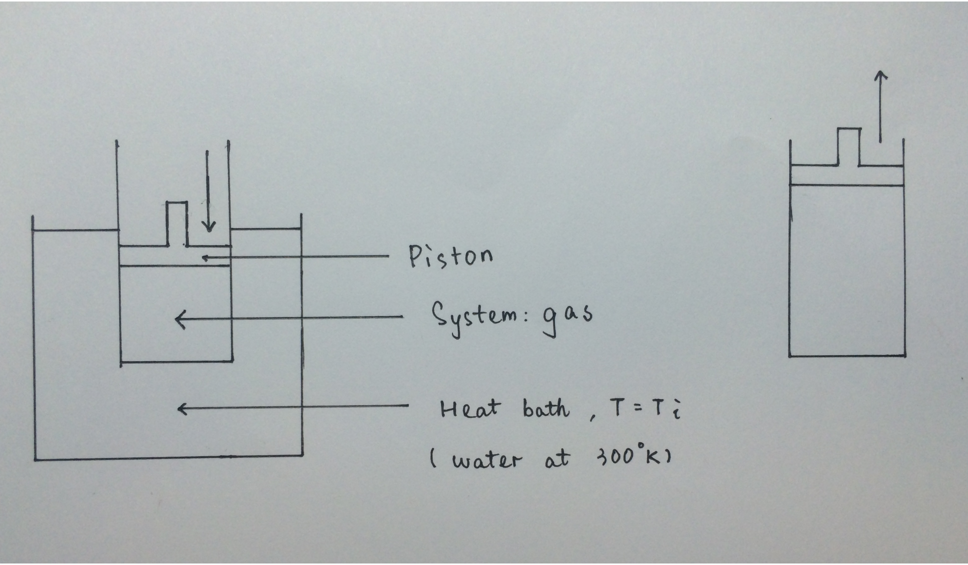
- So $\bar{M}$ is a "generalized force" conjugate to the magnetic field. The work done by the sample is $\bar{M}dH$. By applying the second law of thermodynamics to quasi-static process, we have

$$dQ = TdS = dE + dW = dE + pdV + \bar{M}dH.$$

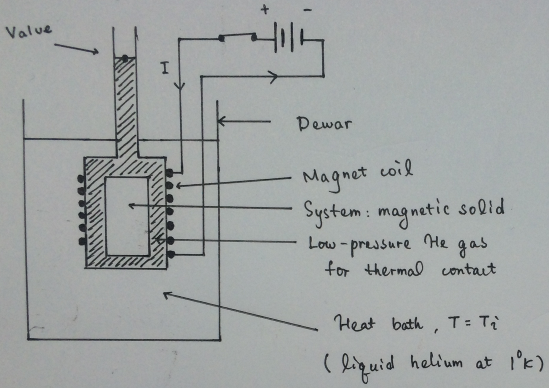
Outline

Mechanical cooling

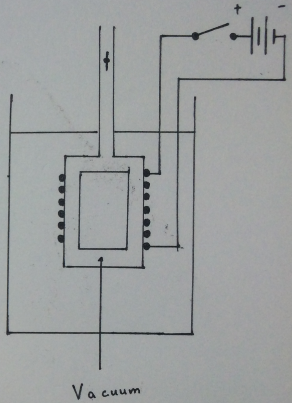


Piston
System: gas
Heat bath , T = Ti
(water at 300°K)

Magnetic cooling

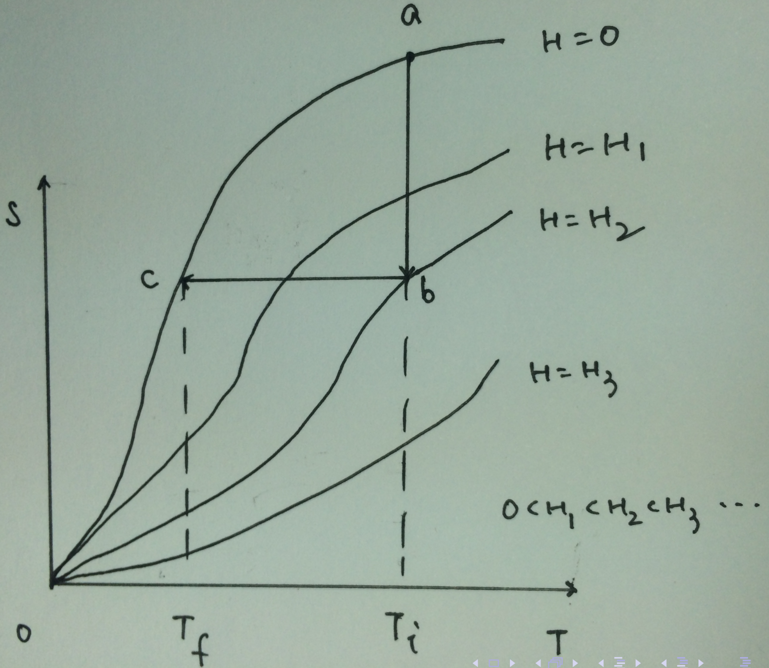


Value
+
-
I
Dewar
Magnet coil
System: magnetic solid
Low-pressure He gas
for thermal contact
Heat bath, T = Ti
(liquid helium at 1°K)



+
-
Vacuum

In fact, the changes in those two processes can be shown in the following figure.


a
H = 0
H = H1
H = H2
H = H3
S
c
b
0 < H1 < H2 < H3 ...
o
Tf
Ti
T

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- Since the entropy is kept constant, it turns to be true that

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Analysis

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- Since the entropy is kept constant, it turns to be true that

$$S(T_f, H_f) = S(T_i, H_i). \tag{3.1}$$

- This calls us in turn to study the temperature's change when S is constant, i.e. $(\partial T/\partial H)_S$, from which we can get T_f by simply integrating over H as long as we know S.

We begin with

$$0 = dS = \left(\frac{\partial S}{\partial T}\right)_H dT + \left(\frac{\partial S}{\partial H}\right)_T dH.$$

So that it is sufficient to know about $(\frac{\partial S}{\partial H})_T$ and $(\frac{\partial S}{\partial T})_H$, since

$$\left(\frac{\partial T}{\partial H}\right)_S = \frac{dT}{dH} = -\frac{\left(\frac{\partial S}{\partial H}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_H}. \tag{3.2}$$

For the numerator

By what mentioned above, $dE = TdS - mdH$, we rewrite this formula in respect with free energy. Then, $dF = -SdT - mdH$. Thus,

$$\left(\frac{\partial S}{\partial H}\right)_T = -\left(\frac{\partial^2 F}{\partial H \partial T}\right) = -\left(\frac{\partial^2 F}{\partial T \partial H}\right) = \left(\frac{\partial m}{\partial T}\right)_H . \tag{3.3}$$

By definition of magnetic moment m, $m = MV = \chi HV$, where M is the magnetization, V the volume and χ the magnetic susceptibility. Since V is a constant here, H and T is two independent variables, so that

$$\left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial m}{\partial T}\right)_H = HV\left(\frac{\partial \chi}{\partial T}\right)_H . \tag{3.4}$$

For the denominator

Define

$$C_H(T,H) = T\left(\frac{\partial S}{\partial T}\right)_H. \tag{3.5}$$

By (3.3), we have

$$\frac{1}{T}\left(\frac{\partial C_H}{\partial H}\right)_T = \frac{\partial^2 S}{\partial H \partial T} = \frac{\partial^2 S}{\partial T \partial H} = \left(\frac{\partial^2 m}{\partial T^2}\right)_H = HV\left(\frac{\partial^2 \chi}{\partial T^2}\right)_H.$$

Integrate over 0 to H, we have

$$\left(\frac{\partial S}{\partial T}\right)_H = \frac{C_H(T,H)}{T} = \frac{C_H(T,0)}{T} + V\int_0^H \frac{\partial^2 \chi(T,H')}{\partial T^2} H' dH'.$$

Summary for this subsection

A knowledge of $C_H(T, 0)$ in zero magnetic field and a knowledge of $\chi(T, H)$ is sufficient to find $(\partial T/\partial H)_S$.

Acknowledgement

Thank you for your attention!